Check if a vector field is conservative

Monday, March 29, 2021 4:15 PM

$$F = \langle P, Q \rangle$$

How to check if F is conservative?

There are two methods:

① Set $f_x = P$, $f_y = Q$ and try to find f.

Ex $F = \langle 2xy, x^2 \rangle$

$$f_x = 2xy \rightsquigarrow f = x^2y + C(y)$$

$$f_y = x^2 + C'(y) \rightsquigarrow C'(y) = 0 \rightsquigarrow \underline{C(y) = 0}.$$

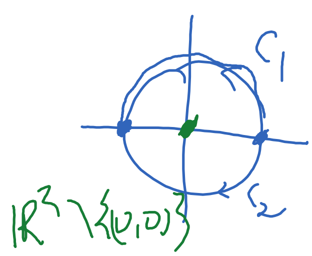
$$f(x,y) = x^2y.$$

② Observation: if $F = \langle P, Q \rangle = \langle f_x, f_y \rangle$ then $P_y = Q_x$.

$$\boxed{\text{If } P_y \neq Q_x \text{ then } F \text{ is } \underline{\text{not}} \text{ a conservative vector field.}}$$

It is, in general, not true that $P_y = Q_x$ implies F is a conservative vector field.

Ex $F = \left\langle \underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q} \right\rangle$

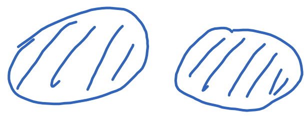


$\underbrace{\int_{C_1} F \cdot dr}$ $\neq$ $\underbrace{\int_{C_2} F \cdot dr}$

$$\begin{cases} P_y = Q_x \text{ in } D \\ D \text{ is a simply connected domain} \end{cases} \Rightarrow F \text{ is conservative.}$$

* Simply connected domain : no holes, no separated pieces.

→  not simply connected

→  not simply connected

 simply conn.

Ex 1) $F(x,y) = \langle \underbrace{xy + y^2}_{P}, \underbrace{x^2 + 2xy}_{Q} \rangle \rightarrow$ not cons.

$$\left.\begin{aligned} P_y &= x + 2y \\ Q_x &= 2x + 2y \end{aligned}\right\} \text{not equal}$$

2) $F(x,y) = \langle \underbrace{y^2 - 2x}_{P}, \underbrace{2xy}_{Q} \rangle$ $\qquad \mathbb{R}^2$: simply conn.

$$\left.\begin{aligned} P_y &= 2y \\ Q_x &= 2y \end{aligned}\right\} \text{equal}$$

F is conservative.